23/11/20
(Με Γραφίδα).

Ασκήσεις:

Πx **Παράδειγμα 3, σελ. 209:**

$y_1' = y_1 - y_2 - y_3$

$y_2' = y_1 + 3y_2 + y_3$

$y_3' = -3y_1 + y_2 - y_3$

Λύση: Παραγωγίζω:

$y_1'' = y_1' - y_2' - y_3' = (y_1 - y_2 - y_3) - (y_1 + 3y_2 + y_3) - (-3y_1 + y_2 - y_3) = 3y_1 - 5y_2 - y_3$

$y_1''' = 3y_1' - 5y_2' - y_3' = 3(y_1 - y_2 - y_3) - 5(y_1 + 3y_2 + y_3) - (-3y_1 + y_2 - y_3) = y_1 - 19y_2 - 7y_3.$

$$\Rightarrow \begin{cases} y_1' = y_1 - y_2 - y_3 & (-5)\ (-19) \\ y_1'' = 3y_1 - 5y_2 - y_3 \\ y_1''' = y_1 - 19y_2 - 7y_3 \end{cases} \rightarrow \boxed{y_3} \Rightarrow \begin{cases} y_1' = y_1 - y_2 - y_3 \\ y_1'' - 5y_1' = -2y_1 + 4y_3 & (-3) \\ y_1''' - 19y_1' = -18y_1 + 12y_3 \end{cases} \Rightarrow$$

$$\Rightarrow y_1''' - 3(y_1'' - 5y_1') - 19y_1' = -12y_1$$

$$\Rightarrow y_1''' - 3y_1'' - 4y_1' + 12y_1 = 0 \rightarrow \boxed{y_1}$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$\begin{matrix} y_1 - y_2 - y_3 = y_1' \\ 3y_1 - 5y_2 - y_3 = y_1'' \\ y_1 - 19y_2 - 7y_3 = y_1''' \end{matrix} \Rightarrow D = \begin{vmatrix} 1 & -1 & -1 \\ 3 & -5 & -1 \\ 1 & -19 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2 & +2 \\ 1 & -18 & -6 \end{vmatrix} = 12 + 36 = 48$$

είναι σωστό; (να ρωτήσω)

$$D_1 = \begin{vmatrix} y_1' & -1 & -1 \\ y_1'' & -5 & -1 \\ y_1''' & -19 & -7 \end{vmatrix} = y_1'(35-19) - y_1''(7-19) + y_1'''(1-5) = 16y_1' + 12y_1'' - 4y_1'''$$

(απλοποιώ με το 4)

$$y_1 = \frac{D_1}{D} = \frac{16y_1' + 12y_1'' - 4y_1'''}{48} = \frac{4y_1' + 3y_1'' - 1y_1'''}{12}$$

$$12y_1 = 4y_1' + 3y_1'' - y_1''' \Leftrightarrow y_1''' - 3y_1'' - 4y_1' + 12y_1 = 0$$

(Πχ) <u>Άσκηση 1iii σελ (i)</u>: Να επιλυθεί το π.α.τ.:

$y_1' = 3y_1 - 2y_2$

$y_2' = -2y_1$ με $y_1(0)=2,\ y_2(0)=1=y_3(0)$.

$y_3' = y_1 + 2y_3$

<u>Λύση:</u> $y_1'' = 3y_1' - 2y_2' = 3y_1' + 4y_1$

$\Rightarrow y_1'' - 3y_1' - 4y_1 = 0$

<u>Χαρ. πολυώνυμο:</u> $\lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = -1.$

$\rightarrow y_1(t) = c_1 e^{4t} + c_2 e^{-t}$

$t=0: y_1(0) = c_1 \cdot 1 + c_2 \cdot 1 \Rightarrow c_1 + c_2 = 2.\ (1).$

$y_2 = -\frac{1}{2} y_1' + \frac{3}{2} y_1.$

$y_2(t) = -\frac{c_1}{2} e^{4t} + 2c_2 e^{-t}$

$t=0: y_2(0) = -\frac{c_1}{2} e^{4\cdot 0} + 2 \cdot c_2 \cdot e^{-0} = -\frac{c_1}{2} + 2c_2 = 1\ (2).$

Από (1),(2): $c_1 = 6/5,\ c_2 = 4/5.$

$$\Rightarrow \begin{cases} y_1(t) = \frac{6}{5} e^{4t} + \frac{4}{5} e^{-t} \\ y_2(t) = -\frac{3}{5} e^{4t} + \frac{8}{5} e^{-t}. \end{cases}$$

$y_3' = y_1 + 2y_3 \rightarrow y_3' - 2y_3 = y_1 \rightsquigarrow \boxed{y_3}$

$$y_3(t) = e^{-\int_0^t (-2)\,ds} \Big[\underbrace{y_3(0)}_{=1} + \int_0^t \dots \Big]$$

(Πχ) $y_1' = a_{11} y_1 + a_{12} y_2$ με $a_{ij} \in \mathbb{R},\ i,j \in \{1,2\}.$

$y_2' = a_{21} y_1 + a_{22} y_2.$

<u>Λύση:</u> $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \rightarrow \bar{y}'(t) = A\bar{y}(t)$ με $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$e^{\lambda t}\bar{c} = \bar{y}(t)$

$\lambda e^{\lambda t}\bar{c} = A e^{\lambda t}\bar{c}$

$\bar{0} = A\bar{c} - \lambda\bar{c} \Rightarrow (A-\lambda I)\bar{c} = \bar{0}$

λ: ιδιοτιμές του A.

Πχ **Παράδειγμα 1, σελ. 207.**

$y_1' = y_1 + 12y_2$

$y_2' = 3y_1 + y_2.$

Λύση: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \underbrace{\begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix}}_{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 36 = (\lambda-1-6)(\lambda-1+6) \quad \lambda_1 = 7, \quad \lambda_2 = -5$$

$\lambda_1 = -5: (A-\lambda I)\bar{c}_1 = \bar{0}$

$$\begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad c_{11} + 2c_{12} = 0 \Rightarrow c_{11} = 2, \quad c_{12} = 1$$

$$\bar{y}_1(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-5t} \rightsquigarrow \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = \begin{bmatrix} 2e^{-5t} \\ -e^{-5t} \end{bmatrix}$$

$\lambda_2 = 7$:

$$\begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad c_{21} - 2c_{22} = 0 \Rightarrow c_{22} = 1, \quad c_{21} = 2.$$

$$\bar{y}_2(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}$$

$$\bar{y}(t) = c_1\bar{y}_1(t) + c_2\bar{y}_2(t) \Leftrightarrow c_1 e^{-5t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1 = 2c_1 e^{-5t} + 2c_2 e^{7t}$ / $\lambda_1 = \lambda_2 = \lambda$

$y_2 = -c_1 e^{-5t} + c_2 e^{7t}$ / λ_1, λ_2: μιγαδικές.

Πx

Άσκηση: $y^{(4)} + y = 0$

Λύση: $\lambda^4 + 1 = 0 \Rightarrow \underbrace{\lambda^4 + 2\lambda^2 + 1} - 2\lambda^2 = 0$

$\Rightarrow (\lambda^2+1)^2 - (\sqrt{2}\lambda)^2 = 0$

$\Rightarrow (\lambda^2 - \sqrt{2}\lambda + 1)(\lambda^2 + \sqrt{2}\lambda + 1) = 0$

$\downarrow \qquad \downarrow$

$\lambda_{1,2} = \dfrac{\sqrt{2} \pm \sqrt{2}\, i}{2} \qquad \lambda_{3,4} = \dfrac{-\sqrt{2} \pm i\sqrt{2}}{2}$

$e^{\pm\frac{\sqrt{2}}{2}t} \quad \cos\frac{\sqrt{2}}{2}t$

$\sin\frac{\sqrt{2}}{2}t$

Πx

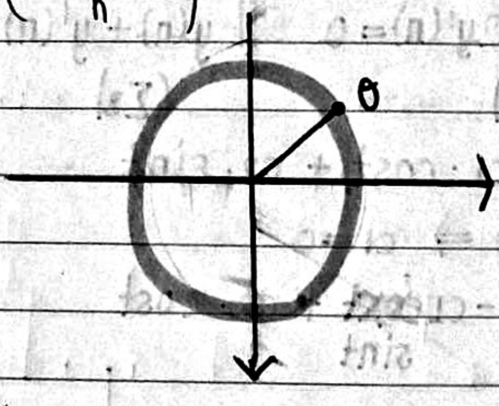
$z^n = z_0 = |z_0|(\cos\theta + i\sin\theta)$

$z_k = \sqrt[n]{|z_0|}\ \cos\left(\dfrac{2k\pi + \theta}{n}\right) + i\sin\left(\dfrac{2k\pi+\theta}{n}\right)$, $k = 0, \dots, n-1$.

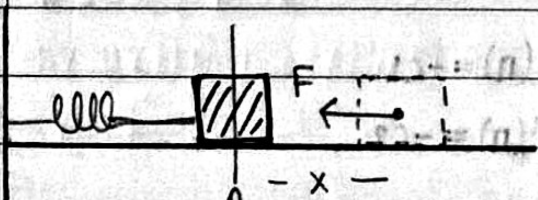
$z_4 = -1 = \cos 0°$

$z_{1,2,3} = \cos\dfrac{2k\pi}{4} + i\sin\dfrac{2k\pi}{4}$

$k = 0, 1, 2, 3.$



Πx



$F = k\cdot x(t)$

$m\gamma = -kx(t)$

$mx''(t) = -kx(t)$

$x''(t) + \dfrac{k}{m}x(t) = 0$

$\lambda^2 + \dfrac{k}{m} = 0$

$\hookrightarrow \lambda = \pm i\sqrt{\dfrac{k}{m}} \Rightarrow \lambda(t) = c_1\cos\left(\sqrt{\dfrac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\dfrac{k}{m}}\,t\right)$

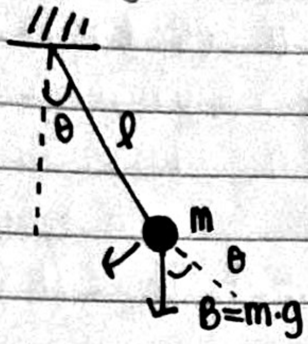
$\lambda'(t) = \left[-c_1\sin\left(\sqrt{\dfrac{k}{m}}\,t\right) + c_2\cos\left(\sqrt{\dfrac{k}{m}}\,t\right)\right]\sqrt{\dfrac{k}{m}}$

$x(0)=x_0 \Rightarrow x(0)=c_1$

$x'(0)=0 \Rightarrow c_2=0$

$\hookrightarrow x(t)=x(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

Πx **Εφαρμογή:**



$F=B\cdot\sin\theta=m\cdot g\cdot\sin\theta=m\cdot\gamma \quad , \gamma=g\cdot\sin\theta.$

$$\boxed{x''=g\cdot\frac{x}{l}}$$

Πx $y'+y=0$ με $y(0)$, $y'(0)$.

$y(0)+y'(0)=0$	$y(0)=y'(0)=0$
$y(\pi)+2y'(\pi)=0$	$y(\pi)+y'(\pi)=0$
(Σ_1)	(Σ_2)

$y(t)=c_1\cdot\cos t+c_2\cdot\sin t$

$y(0)=0 \Rightarrow c_1=0$

$y'(t)=-c_1\cancel{\cos t}\,\sin t+c_2\cdot\cos t$

$\Rightarrow y(t)=c_1\cdot\cos t+c_2\cdot\sin t \quad | \quad y(0)=c_1,\ y(\pi)=-c_1.$

$y'(t)=-c_1\cdot\sin t+c_2\cdot\cos t \quad | \quad y'(0)=c_2,\ y'(\pi)=-c_2.$

(Σ_1): $c_1+c_2=0$

$\hookrightarrow -c_1+2c_2=0$

$c_1=c_2=0$

(Σ_2): $-c_1-c_2=0 \leftarrow [y(0)+y'(0)=0]$

$c_2=-c_1$

$y(\pi)=c_1\cdot\cos t-c_1\cdot\sin t$